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On the Existence of Real Entire Functions with a Prescribed Ordered Set of Stationary Values

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To each ordered set of real stationary values (with multiplicities) is shown to correspond a real entire function f taking on the stationary values in the given order and with the given multiplicities along the real axis. This function has a derivative f', whose roots are all real, and which is of the Pólya–Laguerre class; the function f is, apart from an arbitrary real affine transformation (conserving order) of the independent variable, the only function satisfying all of the conditions above. © 1993 Academic Press, Inc.

INTRODUCTION

Let $f(z)$ be a real entire non-constant function with the property that its derivative $f'(z)$ has all its zeros real. Let the set of zeros x_j of f' be ordered so that $j < j+1$ implies $x_j \leqslant x_{j+1}$. The set of suffixes j is a subset of $\mathbb{Z}$. Here we account for a zero of multiplicity m by ascribing to it a set of m consecutive suffixes. We order the set of stationary values $f(x_j)$ after the value of j. If $f(x)$ has limits (possibly infinite) for $x \to \pm\infty$, these are included among the stationary values in the natural way (but without multiplicities). Then the question to be treated here is whether this ordered set of stationary values determines the function f.

The first thing one could observe is that we can replace $f(z)$ by $g(z) = f(c_1 z + c_2)$, where c_1 and c_2 are real numbers ($c_1 > 0$), without changing the ordered set of stationary values.

Next, it is easy to give examples of pairs (f, g) of functions of different order, but with the same ordered set of stationary values. In fact, let P be a real polynomial of the form $P(z) = a + c \int_0^z \prod_{j=1}^{2n-1} (w - x_j)\, dw$, where a, c, and the x_j ($j = 1, 2, ..., 2n-1$) are real numbers with $c > 0$. The function $f(z) = \exp(P(z))$ then has the property that f' has the same $2n-1$ real roots as P', and $\lim_{|x| \to \infty} f(x) = +\infty$. But it is possible to find a real polynomial Q of degree $2n$ with the same stationary values as f, taken in the same order (see [4]). However, Q is of order 0, and f is of order $2n$.

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A natural question is whether we can achieve unique correspondence between a function f with the properties described above and its ordered set of stationary values, if we prescribe two points of its graph and limit its order suitably. An affirmative answer is given in this paper; in fact, we do have this unique correspondence if the derivative f' of f belongs to the Pólya–Laguerre class.

Formulation of the problem. Let $S = \{(m_j, y_j) \mid j \in A\}$ be an ordered set of ordered couples where the ordered set A is a non-empty set of consecutive integers, and for each $j \in A$ we have $m_j \in \mathbb{N}$ and $y_j \in \mathbb{R}$, except when j is an extreme element of A: then y_j may assume the values $\pm\infty$, and m_j is irrelevant. We further assume that for all $j \in A$ with $\{j-1, j+1\} \subset A$ we have $(y_{j+1} - y_j)(y_j - y_{j-1})(-1)^{m_j} > 0$. We denote by A° the set obtained from A by removing its extreme elements.

A real entire function f is said to be compatible with S, if the following conditions are satisfied,

(1) $(f')^{-1}(0) = \{x_j \mid j \in A\} \subset \mathbb{R}$. Here A is the ordered set referred to above as part of the definition of S, and the implied ordering of $(f')^{-1}(0)$ shall be that inherited from $\mathbb{R}$. An extended real number $-\infty$ or $+\infty$ is said to belong to $(f')^{-1}(0)$ if A has a minimal or a maximal element, respectively.

(2) For all $j \in A^\circ$ the integer m_j is the multiplicity of x_j as a zero of f', and for all $j \in A$ the extended real number y_j equals $f(x_j)$.

We prove the

THEOREM. *For all ordered sets S of the above description there exists a real entire function f compatible with it. Its derivative has the form*

$$f'(z) = ce^{-az^2 + bz} \prod_{j \in A^\circ} (1 - (z/x_j))^{m_j} e^{m_j z/x_j}, \tag{1}$$

where $a \geqslant 0$, and b and c are real numbers. The function f is uniquely determined by S apart from a transformation $z \mapsto c_1 z + c_2$, where c_1 and c_2 are arbitrary real numbers ($c_1 > 0$), of the independent variable. If, in the expression (1), $a > 0$, then the values of $f(-\infty)$ and $f(+\infty)$ are finite. If A has a minimal (resp. maximal) element, and $f(-\infty)$ (resp. $f(+\infty)$) is infinite, then the canonical product is of genus 0.

The following proof of the theorem is divided into several sections. First we prove a number of lemmas, mostly concerning level lines for modulus and argument of entire functions. Next, we prove uniqueness of the function corresponding to an ordered set of stationary values, and finally we give the proof of existence of such a function.

On level lines. Let f be a non-constant function holomorphic in a certain region Ω of the complex plane $\mathbb{C}$ (notation: $f \in H(\Omega)$). Then the associated real function $|f|$ is defined everywhere in Ω. A level line for $|f|$ is a connected set of points in Ω on which $|f|$ is constant. If we except from Ω the zeros of f, we can also define $\arg f$ everywhere, but in general only mod 2π. In the later applications we will be able to remove from Ω a curve from each zero of f in Ω to the boundary $\partial\Omega$ in such a way that the resulting region Ω' is simply connected; then it is possible to define $\arg f$ everywhere in Ω as a continuous function. It is still meaningful, however, for simply connected subsets of the original region Ω, to speak of increases of $\arg f$ along a given curve, and of level lines for $\arg f$ (we shall here not continue a level line for $\arg f$ beyond a zero z_1 of f but may consider z_1 as an endpoint for the level line). The level lines for $|f|$ and $\arg f$ can also be regarded as level lines for the real and imaginary parts of $\log f$. Let z_0 be a point on a level line L for $|f|$ or $\arg f$. If $f'(z_0)$ and $f(z_0)$ are non-zero, L is a simple smooth curve in a neighbourhood of z_0. If $f(z_0) = 0$, the point z_0 is (in Ω') the endpoint of level lines for $\arg f$ corresponding to values in an open interval of length $2m\pi$, where m is the multiplicity of the zero z_0. If $f(z_0) \neq 0$, but z_0 is a zero of f' of multiplicity $m \geqslant 1$ (in the following, a number like z_0 is just called a zero of f' without explicit mention of the condition that $f(z_0) \neq 0$), then we have an expansion $f(z) = f(z_0) + (z - z_0)^{m+1} f_1(z)$ with $f_1 \in H(\Omega)$ and $f_1(z_0) \neq 0$. Let the function f_2 (holomorphic in a neighbourhood of z_0) be defined such as to satisfy the equation $f_2^{m+1}(z) = (f_1(z)/f(z_0))$. Then a level line $|f(z)| = |f(z_0)|$ with z_0 as an endpoint has close to z_0 a parametric equation $z = z(\theta)$, given by

$$((z - z_0) f_2(z))^{m+1} = e^{i\theta} - 1 \tag{2}$$

for $\theta \in I$, where $I \subset \mathbb{R}$ is an interval with $\theta = 0$ as an endpoint. We see that there are in fact $2(m+1)$ level lines for $|f|$ with z_0 as an endpoint, depending on what branch we choose for the $(m+1)$st root of the right-hand side of Eq. (2), and whether I is a left or a right neighourhood of 0. If we move along a small circle with center at z_0, we will meet the level line for $|f|$ in such an order that those where $\arg f$ increases away from z_0 alternate with those where it decreases. Similarly, a parametric equation $z = z(t)$ for a level line $\arg f = \arg f(z_0)$ with z_0 as an endpoint can be found by solving the equation

$$((z - z_0) f_2(z))^{m+1} = t \tag{3}$$

for $t \in I$, where I is an interval of the type described above. Again there are $2(m+1)$ possibilities, and we have alternation with respect to whether $|f|$ increases or decreases along the level line. Furthermore, the two sorts of level lines alternate, when we go round z_0.

The first two lemmas are well known.

LEMMA 1 [7, P. III, No. 191]. *Along a level line for the absolute value* $|f|$ *(resp. the argument* $\arg f$*), on which neither the function* f *nor its derivative* f' *becomes* 0, $\arg f$ *(resp.* $|f|$*) changes monotonically.*

Proof. With the given assumptions the function $\log f$ is defined and differentiable with a non-zero differential quotient in a neighbourhood of each point on the level line. At a point of a level line for $|f|$ the real part of $\log f$ has a zero directional derivative along the level line, and therefore the imaginary part of $\log f$ has a non-zero directional derivative along the curve, which is what we wanted to prove. Similarly for points of a level line for $\arg f$.

LEMMA 2 [7, P. III, No. 192]. *Let* L *be a simple closed level line for* $|f|$. *Assume that the interior* Ω_1 *of* L *is a subset of* Ω.

Then, in Ω_1 *the function* f *has one zero more than its derivative* f'.

Proof. Let $|f| = R$ on L. If f did not have a zero in Ω_1, the Maximum Modulus Theorem used on the two functions f and $1/f$ would provide a contradiction. We also see that in Ω_1 we have $|f| < R$, so that zeros of f' on L can be avoided by choosing a smaller value of R without changing the number of zeros of f or f' inside the curve. Assume this done. In a neighbourhood of each point $z_0 \in L$ we can define the holomorphic function $\theta(z)$ by the equation $f(z) = Re^{i\theta(z)}$, and θ can be continued indefinitely along L, where it is real and equal to a value of $\arg f$. The increase of $\theta(z)$ when z describes L once in the positive direction equals 2π times the number of zeros of f in Ω_1. Similarly, under this movement of z the increase in $\arg(f'/f)(z)$ equals 2π times the difference between the number of zeros of f' and the number of zeros of f in Ω_1. But $f'/f = i\theta'$, and since θ is strictly increasing along L, we can use it as a curve parameter; thus the increase in $\arg(f'/f)(z)$ along L is equal to the decrease of $\arg(dz/d\theta)$, which is -2π since L is a simple closed curve described once in the positive direction; in fact, L is homotopic to a circle, and the homotopy can easily be designed so that the intermediate curves are smooth and the corresponding functions $\arg(dz/d\theta)$ vary continuously.

In the remainder of this section we consider entire functions. First, a lemma whose proof I owe to Professor Bo Kjellberg:

LEMMA 3. *Let* f *be an entire function of finite order* ρ, *and let* $z(t)$ $(0 \leqslant t < \infty)$ *be a curve, satisfying* $|z(t)| \to \infty$ *for* $t \to \infty$, *along which the function* f *is bounded.*

Then, for $\sigma > \rho$, *there is a sequence of values* $t_n \to \infty$, *and a positive constant* A, *so that for all* n *we have*

$$|f'(z(t_n))| \leqslant A\, |z(t_n)|^{\sigma - 1}. \tag{4}$$

If f *is of finite type, we can in* (4) *replace* σ *by* ρ.

Remark. If the stipulation that f should be bounded is dropped, (4) is still valid, if we supply its right-hand side with an extra factor $|f(z(t_n))|$. However, the proof I found is prohibitively complicated, and I have not been able to prove the last statement of the lemma in this case.

Proof. We can assume that f is non-constant. Then the function $M(r) = \max_{|z| \leqslant r} |f(z)|$ increases toward infinity, but there exists a number A, so that $\log(M(r)) < Ar^\sigma$ for $r > R_0$, say. We also assume that $|f(z)| \leqslant 1$ on the given curve. Let $z_n = z(t_n)$ $(n = 1, 2, ...)$ be a sequence of points on the curve, and let the sequence (r_n) $(n = 1, 2, ...)$ of positive numbers have the property that the circles $C(z_n, 2r_n)$ with center z_n and radius $2r_n$ do not overlap. Then Kjellberg shows [3, Theorem 1 and its proof] that the number $\nu(r)$ of circles $C(z_n, 2r_n)$ inside a circle $C(0, r)$, for which $\max |f(z)| > 2$ on the circles $C(z_n, r_n)$, is bounded by the inequality

$$\int_0^r \frac{\nu(t)}{t} dt < 100 \log M(r) \tag{5}$$

for sufficiently large r. It follows from (5) that $\nu(r) < 100 \log M(er)$. We now assume that R_0 is chosen larger than the lower bounds referred to above, and also so large that any circle $|z| = r$ with $r \geqslant R_0$ intersects the given curve. Define, for $m \in \mathbb{N}$, $R_m = R_0 e^m$. As usual we denote, for $x > 0$, by $[x]$ the greatest integer less than or equal to x. Let $\delta_m = (1 - 1/e) R_m^{1-\sigma}/(1000Ae^\sigma)$. Then there is room for $[200A(eR_m)^\sigma]$ non-overlapping circles $C(z_n, 2\delta_m)$ with centers on the given curve between $|z| = R_{m-1}$ and $|z| = R_m$. According to the estimate (5) at least $s_m = [100A(eR_m)^\sigma]$ of these will have $\max |f(z)| \leqslant 2$ on the circle $C(z_n, \delta_m)$ (as $\sigma > 0$, it is always possible to choose R_0 so large that the integer s_m becomes positive for every m). By Cauchy's inequality we have $|f'(z)| \leqslant 2/\delta_m$ at the center of such a circle, which implies the required result.

In the next lemma we specialize to real entire functions f whose derivative f' has the form of Eq. (1). Since f is real, it often suffices to investigate the restriction of f to the open upper halfplane U (or, occasionally, to its closure $\bar{U} = U \cup \mathbb{R}$). Remember that the zeros of f' are real and so outside U. We are particularly interested in level lines which are at the same time asymptotic paths. An asymptotic path for an entire function f is a curve $z(t)$ $(0 \leqslant t \leqslant \infty)$ with $|z(t)| \to \infty$ for $t \to \infty$, along which $f(z(t))$ approaches a finite limit a for $t \to \infty$. Here a is a so-called asymptotic value, and there are at most 2ρ of these, where ρ is the order of f (see, for instance, [2, Chap. VIII, Sect. 5]). In our case ($\rho \leqslant 2$, thus finite) we can avoid an asymptotic value 0 by adding a suitable fixed real constant to the numbers y_j $(j \in A)$. We have

LEMMA 4. *Let f be a real entire function, whose derivative f' has the form of Eq. (1). Assume that f does not have 0 as an asymptotic value. Let $z(t)=x(t)+iy(t)$ $(0 \leqslant t < \infty)$ be the parametric equation for a level line L in U with $|z(t)| \to \infty$ for $t \to \infty$, and assume that L is also an asymptotic path.*

Then either $x(t) \to \infty$ or $x(t) \to -\infty$ for $t \to \infty$, and every curve $z_1(t)$ $(0 \leqslant t \leqslant \infty)$ with $|z_1(t)| \to \infty$ for $t \to \infty$, situated below L, is an asymptotic path corresponding to the same asymptotic value (which thus equals either $f(-\infty)$ or $f(\infty)$).

Proof. Clearly, $|f|$ is bounded along L, and Lemma 3 is applicable with $\sigma = 2$. If $|x(t)|$ were bounded along L, there would exist a fixed constant A and a sequence of points $(x_n + iy_n)$ on L, so that $y_n \to \infty$ for $n \to \infty$, and so that $|f'(x_n + iy_n)| \leqslant Ay_n$ for all n. But from Eq. (1) it follows that

$$|f'(x+iy)| = |c|\, e^{a(y^2-x^2)+bx} \prod_{j \in A^\circ} \left(\left(1-\frac{x}{x_j}\right)^2 + \left(\frac{y}{x_j}\right)^2\right)^{m_j/2} e^{m_j x/x_j} \tag{6}$$

for all real x and y. This results in a contradiction, unless $a=0$, and A° has at most one member. If A° has exactly one member (we must then even have $m_1 = 1$), and $a=0$, we can integrate Eq. (1), which gives us an expression for f of the form $f(z)=k+c_1(z-z_0)\exp(b_1 z)$. But clearly, f cannot then be bounded along a curve with bounded abscissa. If A° is empty and $a=0$, f has the form $f(z)=k+c_1 e^{bz}$, and no curve with bounded abscissa can be an asymptotic path.

We have shown that $|x(t)|$ is unbounded along L. The proof that $x(t)$ approaches either $+\infty$ or $-\infty$ depends on the following development.

First we show that the image $f(\mathscr{S})$ of the set $\mathscr{S}$ of points under the curve L is bounded.

We can assume that $x(t)$ takes on arbitrary large values. Let $z_2 = x_2 + iy_2$ be an arbitrary point in U under L, i.e., there is a parameter value $t = t_2$ with $x(t_2) = x_2$ and $y(t_2) > y_2$. Define the curve L_2 by $z_2(t) = x(t) + iy(t)\, y_2/y(t_2)$ $(0 \leqslant t \leqslant t_2)$, where t is the parameter for L (for definiteness we can let t be the arc length along L). Then

$$f(z_2) = f(z_2(0)) + \int_0^{t_2} f'(z_2(t))\, z_2'(t)\, dt. \tag{7}$$

Note that for $0 \leqslant t \leqslant t_2$, we clearly have $|z_2'(t)| \leqslant |z'(t)|$, and also, because of Eq. (6), $|f'(z_2(t))| \leqslant |f'(z(t))|$.

If L is a level curve $|f| = R$, we can, in a neighbourhood of L, define a holomorphic function $\theta(z)$ by $f(z) = Re^{i\theta(z)}$. Then $f'(z) = i\theta'(z) f(z)$. Along L we have $(d/dt)\log f(z(t)) = i(d/dt)\,\theta(z(t))$, where, according to Lemma 1, $(d/dt)\,\theta(z(t))$ is a real function of constant sign. Thus, using the observations above, we obtain

$$\begin{aligned}|f(z_2)-f(z_2(0))| &\leqslant \int_0^{t_2} |f'(z_2(t))|\,|z_2'(t)|\,dt\\ &\leqslant \int_0^{t_2} |f'(z(t))|\,|z'(t)|\,dt\\ &= \int_0^{t_2} |f(z(t))|\,|\theta'(z(t))\,z'(t)|\,dt\\ &= R\int_0^{t_2}\left|\frac{d}{dt}\,\theta(z(t))\right|dt\\ &= R\left|\int_0^{t_2}\frac{d}{dt}\,\theta(z(t))\,dt\right|\\ &= R\,|\theta(z(t_2))-\theta(z(0))|,\end{aligned} \tag{8}$$

which is bounded independently of t_2, since the argument variation along the asymptotic path L is finite. Thus $f(\mathscr{S})$ is bounded. A bound is $\max|f|\,(C)+R\,|\Delta\theta|$, where C is the segment connecting the two points $x(0)$ and $z(0)$, and $\Delta\theta$ is the difference between the initial value of θ and its limit for $t\to\infty$ along L.

Actually, we can give an exact bound for $f(\mathscr{S})$: Let $z_3\in U$ be a point outside the closure of the union of $\mathscr{S}$ and its mirror image with respect to $\mathbb{R}$. Consider for small positive values of ε a function $g_\varepsilon(z)=f(z)/(z-z_3)^\varepsilon$. Use the Maximum Modulus Theorem on the function g_ε for $\varepsilon\to 0$ to show that the least upper bound for $|f|$ in $\mathscr{S}$ is $\max\{\max|f|\,(C), R\}$.

For the case where L is a level line $\arg f=\theta$, we can assume that L does not contain any zero of f except possibly as an endpoint. Then we define, in a neighbourhood of L, the holomorphic function $R(z)$ by $f(z)=R(z)\,e^{i\theta}$. So $f'(z)=R'(z)\,e^{i\theta}$. Along L we have $(d/dt)\,f(z(t))=e^{i\theta}(d/dt)\,R(z(t))$, where $(d/dt)\,R(z(t))$ is a real function without zeros on L. Hence, as in Eq. (8),

$$\begin{aligned}|f(z_2)-f(z_2(0))| &\leqslant \int_0^{t_2} |f'(z(t))|\,|z'(t)|\,dt\\ &= \int_0^{t_2} |R'(z(t))\,z'(t)|\,dt\\ &= \int_0^{t_2}\left|\frac{d}{dt}\,R(z(t))\right|dt\\ &= \left|\int_0^{t_2}\frac{d}{dt}\,R(z(t))\,dt\right|\\ &= |R(z(t_2))-R(z(0))|\\ &= |f(z(t_2))-f(z(0))|.\end{aligned} \tag{9}$$

But f is bounded on L, so that we again find that $f(\mathscr{S})$ is bounded. Here we also have $\sup |f|\,(\mathscr{S}) = \sup |f|\,(L \cup C)$.

It follows (see [7, P. III, Nos. 339, 340]; use the reality of f) that any curve L_1 situated under L and with $x_1(t) \to +\infty$ for $t \to \infty$ is an asymptotic path with asymptotic value $f(+\infty)$, which must exist in this case. Similarly, if $x_1(t) \to -\infty$ for $t \to \infty$, the asymptotic value is $f(-\infty)$. Now, $x(t)$ cannot come arbitrarily close to both $+\infty$ and $-\infty$: As $|z(t)| \to \infty$ for $t \to \infty$, to any point $z_0 \in \bar{U}$ we can find a T, so that $|z(t)| > |z_0|$ for $t > T$. For $x(t) = x_0$ this implies $y(t) > y_0$, so that any point in $\bar{U}$ would belong to $\mathscr{S}$, implying that f should be bounded on $\mathbb{C}$ and thus constant, which is not permissible. This concludes the proof of Lemma 4.

Now, for a function f satisfying the conditions of Lemma 4, we can give more details concerning the behaviour of non-finite level lines for $|f|$. In fact, we have

LEMMA 5. *Let f be a real entire function, whose derivative f' has the form of Eq. (1). Let $L \subset U$ be a level line for $|f|$. Assume that L goes toward infinity and is complete, i.e., L is a component of a set $|f| = R$. Then L divides U into two parts, one of which is a component of the set $|f| \leqslant R$. The closure $(\mathscr{S})$ of this component in $\bar{U}$ contains either the whole of $\mathbb{R}$ or a semi-infinite real interval.*

Proof. We first note that the case where L is an symptotic path with its abscissa bounded in one direction was already treated in the proof of Lemma 4. The case where L is the union of two asymptotic paths can be treated in the same way, and we disregard these cases in the following. In other words, we assume that $\arg f$ has infinite variation along L.

Let L be parametrized as $z(t)$ $(t \in I)$, where the open interval I is either $(0, \infty)$ or $(-\infty, +\infty)$, depending on whether L approaches ∞ in one or in both directions (in the first case the closure $\bar{L}$ of L in $\bar{U}$ contains a point x_0 of $\mathbb{R}$). Since $|f|$ is constant along L, but f' is non-zero here, $|f|$ must, at any point of L, be strictly decreasing in some continuously varying direction normal to L, and so L is part of the boundary of a component $\mathscr{S} \subset \bar{U}$ of the set $|f| \leqslant R$. Let $z_0 \in U$ be a point on L with $\arg f(z_0) = \theta_0 \not\equiv 0$ (mod π). Follow the level line $\arg f = \theta_0$ into $\mathscr{S}$. We have seen in Lemma 4 that a corresponding asymptotic value must be real (remember that we have arranged possible asymptotic values to be different from zero); thus, the level line cannot be an asymptotic path and cannot end in a zero for f' either; it must necessarily lead to a zero $z_1 \in \mathscr{S}$ of f.

If z_1 is non-real, we consider for c increasing from zero the component K_c of the set $|f| \leqslant c$ containing z_1. For small values of c this component contains only the one zero z_1 of f (to see this, we use Lemma 2 and the fact that f' has only real zeros), and the variation of $\arg f$, when the boundary

$|f| = c$ is described once in the positive direction, is 2π. In particular, in the (later to be defined) cut upper halfplane U', where the function $\arg f$ is defined and continuous, only level lines $\arg f = \theta_0$ corresponding to values of θ_0 belonging to an interval of length 2π can end at one particular non-real zero z_1 of f (similarly, if z_1 is a real zero of f of multiplicity m, only level lines $\arg f = \theta_0$ corresponding to an interval of length $m\pi$ can end at z_1). Thus, the number of zeros of f must be infinite.

Considering again a particular non-real zero z_1 of f, and for values of c increasing from zero the component K_c defined above, we see that there is a least upper bound c_1 to the set of values of c, for which K_c contains only the one zero z_1 of f. Evidently, $c_1 < R$. Since, for any $c > c_1$, K_c contains more than one zero for f, thus at least one zero for f', it must also contain points of $\mathbb{R}$. But K_{c_1} can contain at most one point of $\mathbb{R}$: the union of this component and its mirror image in the lower halfplane is simply connected and therefore contains a whole interval of $\mathbb{R}$ if it contains two points, and this would not be compatible with the minimality of c_1.

If K_{c_1} does not contain any points of $\mathbb{R}$, any K_c, where $c > c_1$, contains a semi-infinite real interval, let us, for definiteness, say that it has $+\infty$ as a right endpoint. Then $|f(+\infty)| = c_1$, and the interior of K_{c_1} contains an asymptotic path, going toward $+\infty$, which is a level line for $\arg f$ starting at z_1. The boundary $|f| = c_1$ of K_{c_1} consists of two asymptotic paths.

Let us again look at the case where the boundary curve $|f| = c_1$ has exactly one point x_2 in common with $\mathbb{R}$. Then $f'(x_2) = 0$ and $f(x_2) \neq 0$. We discussed this situation at the beginning of this section, and the essential thing in the present context is that for each point x_2 there is only a finite number of level lines for $\arg f$ with endpoint at x_2. Exactly one of them goes toward z_1.

Let us consider the points $z_0 \in L$ with $\arg f(z_0) \not\equiv 0 \pmod{\pi}$. From each such point we go, in the manner described, along a level line for $\arg f$ to a zero of f. If this zero is non-real we go along the uniquely determined level line for $\arg f$ on to a zero of f' or to an infinitely far point on $\mathbb{R}$ (i.e., follow an asymptotic path). Zeros of f situated between two such curves from L to $\mathbb{R}$ are also connected to L in this way: $|f|$ increases along a level line for $\arg f = \theta_0$ with $\theta_0 \not\equiv 0 \pmod{\pi}$ from a zero of f, and this level line cannot be an asymptotic path or go toward $\mathbb{R}$, and it cannot cross the other curves from L to $\mathbb{R}$, so it must go toward L.

Similarly, we can consider zeros x_2 of f' between two such curves from L to $\mathbb{R}$. Following a level line L_1 for $\arg f$, along which $|f|$ decreases, from x_2, we must first consider the possibility that L_1 could be an asymptotic path, for definiteness assumed going toward $+\infty$, or go back toward $\mathbb{R}$. Then $|f|$ must also decrease along $\mathbb{R}$ to the right; but between two level lines for $\arg f$ from x_2 along which $|f|$ decreases there must be one along which $|f|$ increases, which gives a contradiction to the results of Lemma 4 or the Maximum Modulus Theorem. Thus L_1 goes to a non-real zero of f, which is again connected to L.

If $|f|$ increases along the given level line L_1 for $\arg f$, L_1 cannot return to $\mathbb{R}$, since its endpoint would be a zero of f', for which the previous argument could be used. If L_1 is an asymptotic path, for definiteness going toward $+\infty$, there must be a level line L_2 (possibly $\mathbb{R}$) for $\arg f$ from x_2 below L_1, along which $|f|$ decreases toward the right. Since there cannot be any zeros for f below L_1 (as earlier said, all their level lines for $\arg f$, except one or two, would go toward L but cannot cross L_1), L_2 must be an asymptotic path along which $|f|$ decreases all the way, but with the same asymptotic value as L_1, an obvious contradiction. Thus L_1 must connect the point x_2 directly to a point of L.

Assume that the curve L is not the whole boundary of $\mathscr{S} \cap U$ in U. Then there must exist another level line $L_1 \subset U$, which is a part of the boundary of $\mathscr{S}$. For definiteness we consider the case where L_1 lies to the right of L. Then the right part L_R of L cannot be an asymptotic path (Lemma 4), and the variation of $\arg f$ along L_R is infinite. Let $C_1 \subset \mathscr{S}$ be an arbitrary curve, containing no zeros of f, leading from a point of $\mathbb{R}$ to a point of L. Then the total variation of $\arg f$ along C_1 is finite (use the fact that $|f'/f|$ is bounded along C_1). Thus points of L_R can be connected, by level lines for $\arg f$, to only a finite number of zeros of f to the left of C_1. Similarly, points of L_R can only be connected in this way to a finite number of zeros of f to the right of a curve C_2, containing no zeros of f, leading from a point of $\mathbb{R}$ to a point of L_1. Then f has an infinite number of non-real zeros connected to L between the two curves C_1 and C_2. These zeros must, however, connect to zeros of f' or to infinitely far points on $\mathbb{R}$, and we again have a contradiction to the finite total variation of $\arg f$ along the curves C_1 and C_2.

This concludes the proof of Lemma 5.

Proof of uniqueness. In [5], a proof of the unique correspondence between S and f, with the one essential restriction that the zeros of the function f itself were supposed to be real, was given. It would be natural to try to extend the method of proof to the present more general case.

Let f and g be two functions with derivatives of the form (1), both compatible with S. As in [5] we investigate the function ϕ, defined locally as $g^{-1} \circ f$, at first defined only in a vicinity of the real axis in such a way as to map $\mathbb{R}$ onto $\mathbb{R}$. This can be done because f and g have the same behaviour at $\pm\infty$, and because the roots of f' can be mapped bijectively onto the roots of g' preserving order and multiplicity. In fact, let us start at a point x_j, which is a zero of f' with multiplicity m_j; the corresponding point $w_j = \phi(x_j)$ is a zero of g' of the same multiplicity, and we have $f(x_j) = g(w_j) = y_j$. In a neighbourhood $\mathscr{N}$ of x_j we can put $f(z) =$

$y_j+(z-x_j)^{m_j+1}f_1(z)$, where f_1 is entire with $f_1(x_j)\neq 0$. Similarly, $g(w)=y_j+(w-w_j)^{m_j+1}g_1(w)$, if w is in a suitable neighbourhood of w_j. We wish to define ϕ so that $g(\phi(z))=f(z)$ in a neighbourhood $\mathcal{N}_1$ of x_j. This means that $(z-x_j)^{m_j+1}f_1(z)=(\phi(z)-w_j)^{m_j+1}g_1(\phi(z))$ in $\mathcal{N}_1$. Choose the sign $s\in\{-1,1\}$ so that sf_1 is positive in a real neighbourhood of x_j (then also sg_1 is positive in a real neighbourhood of w_j). Next we choose the branches $f_2=(sf_1)^{1/(m_j+1)}$ and $g_2=(sg_1)^{1/m_j+1)}$ so that they are positive in real neighbourhoods of x_j and w_j, respectively. The equation $(z-x_j)f_2(z)=(\phi(z)-w_j)\,g_2(\phi(z))$ is obviously, by the Implicit Function Theorem, satisfied by a function $\phi(z)$ holomorphic in a neighbourhood of $z=x_j$. The function ϕ can be uniquely analytically continued along the real axis until a neighbouring zero x_{j-1} or x_{j+1} of f', where the same procedure can be used to define ϕ in a neighbourhood. This process can go on until we have defined ϕ in a neighbourhood of the real axis. Note that ϕ' is positive along $\mathbb{R}$.

We shall show that the function ϕ can be continued analytically to the whole plane (since the functions f and g are real, we consider only their behaviour in $\bar{U}$). Because of the symmetry of the situation the same will then be true for its inverse function ψ, defined locally as $f^{-1}\circ g$. Thus ϕ is of the form $\phi(z)=c_1z+c_2$, where c_1 and c_2 are real numbers (with $c_1>0$), which is what we wanted to show.

First let us go back to the situation in [5], where not only $(f')^{-1}(0)\subset\mathbb{R}$, but also $f^{-1}(0)\subset\mathbb{R}$. Here it is possible to define the function $\log f(z)$ for $z\in U$. This function is even univalent on U, and $(\log f)(U)$ is determined by S, so that ϕ can be defined simply as the biholomorphic function $(\log g)^{-1}\circ(\log f)$. Under our more general assumptions we must still ascertain that the behaviour of f on U is essentially determined by S, i.e., by its behaviour on $\mathbb{R}$.

It is natural to consider two cases:

(a) f is bounded on $\mathbb{R}$. Let $R>\sup|f|\,(\mathbb{R})$. The level line L bounding the component K of the set $|f|\leqslant R$ containing $\mathbb{R}$ is, because of the Maximum Modulus Theorem, non-compact, and according to Lemma 4 the variation of $\arg f$ along it is infinite in both directions.

Assume that f has a zero z_1 outside K. Then z_1 cannot, by means of a level line for $\arg f$, be connected to a zero of f' or to an infinitely far point on $\mathbb{R}$, since this would contradict either Lemma 1 or Lemma 4. But this contradicts the results obtained in the proof of Lemma 5. We conclude that f has no zeros outside K.

Assume next that there is a point z_0 outside K, so that $0<|f(z_0)|\leqslant R$. The component K_0 of the set $|f|\leqslant|f(z_0)|$ containing z_0 cannot have any point in common with K, and K_0 cannot be compact, since this would imply the existence of a zero of f in K_0, which we have ruled out. Thus the boundary L_0 of K_0 is of the type discussed in Lemma 5 (it is clearly not an asymptotic path). But the closure of K_0 in $\bar{U}$ is K_0, and we have a contradiction to Lemma 5.

We conclude that on the set $|f| > \sup |f|$ ($\mathbb{R}$) the function $\log f$ can be defined as a holomorphic function and is univalent. The next step is to try to extend the definition of $\log f$ into the set $|f| \leqslant \sup |f|$ ($\mathbb{R}$). To be able to define $\arg f$ here we remove each level line $\arg f = \theta_1$ leading from a non-real zero z_1 of f either to a zero of f' or to an infinitely far point on $\mathbb{R}$. For each zero z_1 there is exactly one such level line. In fact, every level line $\arg f = \theta_0$ with $\theta_0 \not\equiv 0 \pmod{\pi}$ must cross the level line L discussed above, and so must also the level line $\arg f = \theta_0$ with $\theta_0 \equiv \theta_1 + \pi \pmod{2\pi}$, since two level lines for $\arg f$ can only meet at a zero for either f or f'. The set remaining when we have removed these curves from U is called U'. Clearly U' is simply connected and contains no roots of f or f', so $\log f$ can be defined.

Next, we show that the defined function $\log f$ is univalent in U'. We know that along L the function $\arg f$ is monotone with range $\mathbb{R}$. From each point $z_0 \in L$ we follow the level line $\arg f = \theta_0$ into K. If $\theta_0 \not\equiv 0 \pmod{\pi}$, this curve ends in a real or a non-real zero for f. It is easily seen that the set of level lines $\arg f = \theta_0$ ending at a particular non-real zero z_1 of f corresponds to an interval $\theta_1 < \theta_0 < \theta_1 + 2\pi$. Each of the level lines $\arg f = \theta_1$ and $\arg f = \theta_1 + 2\pi$ from points on L either ends at a zero of f' on $\mathbb{R}$ or is an asymptotic path. Similarly, the set of level lines $\arg f = \theta_0$ ending at a particular real zero x_1 with multiplicity m of f corresponds to an open interval $\theta_1 < \theta_0 < \theta_1 + m\pi$, while each of the two level lines $\arg f = \theta_1$ and $\arg f = \theta_1 + m\pi$ either ends at a zero of f' or is an asymptotic path.

It is clear that if a level line $\arg f = \theta_2$ is an asymptotic path, for definiteness going toward $+\infty$, then of the level lines $\arg f = \theta_0$ with $\theta_0 < \theta_2$ (note that $\arg f$ decreases when we move to the right along L) those of the type $\theta_0 = \theta_1$ in the notation from above will be asymptotic paths, while the rest go to non-real zeros of f. Thus, in this case, the set of zeros of f' is bounded above, and the set A has a maximal element.

Now let z_0 be an arbitrary point in $K \cap U'$. Then the level line $\arg f = \arg f(z_0)$ will, in the case where $\arg f(z_0) \not\equiv 0 \pmod{\pi}$, lead to L in the direction of increasing $|f|$, so that z_0 is uniquely characterized by $\log f(z_0)$. But in the case where $\arg f(z_0) \equiv 0 \pmod{\pi}$ the level line $\arg f = \arg f(z_0)$ is squeezed between the level lines for $\arg f$ through the neighbouring points on the level line $|f| = |f(z_0)|$ and so also must cross L. Here we have used the continuity of $\log f$ on U'. So $\log f$ is univalent on U'.

It is also clear that the image $\Omega = \log f(U')$ is uniquely determined (except for an additive constant, which is a multiple of $2\pi i$) by the set S. In fact, Ω is the whole plane with horizontal cuts for each level line $\arg f = \theta_1$ connecting L with a (possibly infinitely far) point on $\mathbb{R}$. Then the cut is placed at $y=\theta_1$ and stretches from $x=-\infty$ to $x=\log|y_1|$, where y_1 is the corresponding stationary value. The vertical distance between two cuts is 2π, if there is a non-real zero of f between the two level lines, and $m\pi$, if there is a real zero of f with multiplicity m between the level lines. It is not true, however, that the image Ω determines the set S (we cannot see the difference between a real double root an a non-real root). Nevertheless, for the function g we can define a cut open upper halfplane V' on which $\log g$ is defined and univalent, and we can, possibly in more than one way, normalize the image $\log g(V')$ to equal Ω. Then the function ϕ can be defined as $(\log g)^{-1} \circ \log f$. But the knowledge we have not been able to transfer from S to Ω is the relative position of the cuts we applied to U to obtain U'. This means that to each non-real zero of f we can associate two distinguished points on the corresponding sides of two cuts in Ω, both with abscissa equal to the logarithm of the absolute value of the associated stationary value. The image plane of $\log g$ can be normalized so that distinguished points in this plane and in Ω correspond. So the function ϕ now has the property that it can be extended to $\bar{U}$ by continuity so that its image equals $\bar{V}$, the halfplane where g is defined. Clearly, this definition of ϕ reduces to the one we started with in a neighbourhood of $\mathbb{R}$.

(b) The function f is unbounded on $\mathbb{R}$. We have the following two subcases:

(b1) $f(x)$ is bounded (by B, say) when x approaches either $+\infty$ or $-\infty$; let us for definiteness consider the first possibility. Choose a point $x_0 \in \mathbb{R}$ with $f'(x_0)<0$, with $|f(x_0)|>B$, and with $|f(x_0)|>|f(x)|$ for all $x>x_0$. Then the level line L for $|f|$ through x_0 cannot be compact, and it cannot be an asymptotic path. We can then use Lemma 5, which implies that $|f|<|f(x_0)|$ to the right of L, and that the number of zeros of f here is infinite. From each non-real zero of f there is, as we noted in the proof of Lemma 5, a unique level line for $\arg f$ going toward $\mathbb{R}$. We remove this level line. But we can find a sequence (x_n) of values satisfying the conditions laid on x_0 above, so that $x_n \to -\infty$ for $n \to \infty$. Then any point of U (in particular any non-real zero of f) lies to the right of some line L_n, defined as the level line for $|f|$ through x_n. For each non-real zero we remove its level line for $\arg f$ leading toward $\mathbb{R}$. In the resulting simply connected region U' we can define $\log f$ as a holomorphic univalent function and complete the proof as before (note that in this case the image $\Omega=\log f(U')$ may be bounded above).

(b2) $|f(x)|$ is unbounded both for $x \to +\infty$ and for $x \to -\infty$. Starting from an arbitrary point $x_0 \in \mathbb{R}$ we define a sequence (x_n) of real numbers going toward $-\infty$ and satisfying the conditions that for each $n \in \mathbb{N}$ we have $x_n < x_{n-1}$, $f'(x_n)<0$, and $|f(x_n)|>|f(x)|$ for $x_n < x \leqslant x_{n-1}$. The corresponding level lines L_n are now compact (otherwise we would have a contradiction to Lemma 5), i.e., they go back to $\mathbb{R}$. Here the analogue of Lemma 5 is evident, and we can again remove level lines for $\arg f$ from non-real zeros of f to $\mathbb{R}$. The rest of the proof is as before (in this case $\Omega = \log f(U')$ may be bounded above or below).

This concludes the proof of the uniqueness part of the theorem.

Proof of existenxce. For the proof that to each set S there exists a real entire function with derivative of the form of Eq. (1) compatible with it, it is natural to consider several cases:

(1) The set A, occurring in the definition of S, is finite. Only the ordinal of A is of interest, and we can set $A = \{-1, 0, 1, ..., n+1\}$. Then we may replace Eq. (1) by

$$f'(z) = ce^{-az^2 - bz} \prod_{k=0}^{n} (z - x_k)^{m_k}. \tag{10}$$

We make use of the two free parameters c_1 and c_2 mentioned in the theorem to normalize a possible function f by stipulating that $x_0 = 0$ and $x_n = 1$.

There are two subcases, depending on the values of y_{-1} and y_{n+1}:

(1a) Both y_{-1} and y_{n+1} are infinite. This subcase was treated in [4].

(1b) Let y_{n+1}, say, be finite. To show existence of a function f we use the method and results of [4]. The equations to be solved are

$$\int_{x_{j-1}}^{x_j} f'(x)\, dx = y_j - y_{j-1} \qquad (j_{\min} \leqslant j \leqslant n+1), \tag{11}$$

where $j_{\min}$ is 0, if y_{-1} is finite (then $a > 0$), and 1 otherwise. Because of the sign restrictions on the right-hand sides of Eqs. (11) we can replace both the integrands and the right-hand sides with their absolute values. We note that the results actually proved in [4] are based on assumptions weaker than those mentioned above. In fact, unique solvability of Eqs. (11) for $1 \leqslant j \leqslant n$ with $\alpha = b = 0$ was proved for the case where an extra factor, a non-negative weightfunction $w_n(x)$ holomorphic in x and possibly in a number of parameters, was allowed in the integrands. In our case the factor $\exp(-ax^2 - bx)$ of Eq. (10) is such a weightfactor. For each pair of values (a, b) we can then solve for the unknowns $c, x_1, ..., x_{n-1}$, which will become holomorphic functions of a and b. Let

$$I_j = \int_{x_{j-1}}^{x_j} |f'(x)|\, dx \qquad (j_{\min} \leqslant j \leqslant n+1) \tag{12}$$

and put

$$I_S = I_1 + I_2 + \cdots + I_n. \tag{13}$$

Put

$$Y_j = |y_j - y_{j-1}| \qquad (j_{\min} \leqslant j \leqslant n+1) \tag{14}$$

and

$$Y_S = Y_1 + Y_2 + \cdots + Y_n. \tag{15}$$

We consider separately the two possibilities:

(1b1) If y_{-1} is infinite, we have $\alpha = 0$ and $b > 0$ in Eq. (10). We see that the ratio I_{n+1}/I_S approaches infinity when $b \to 0$ and approaches 0 when $b \to \infty$ (the latter is easier to see when numerator and denominator are multiplied by e^b). Thus the ratio takes the value Y_{n+1}/Y_S at least once. That there is exactly one solution follows from the uniqueness proof given earlier.

(1b2) If y_{-1} is finite ($a > 0$), a more complicated argument is necessary. For each positive value of a we see as above that there is at least one value b_1 of b, for which $I_0 = Y_0$, and at least one value b_2 of b, for which $I_{n+1} = Y_{n+1}$. We shall need the fact that b_1 and b_2 are uniquely determined continuous functions of a. To prove this rewrite the equations to be solved as

$$F(v) = Y, \tag{16}$$

where the vector Y has the coordinates Y_j ($0 \leqslant j \leqslant n+1$), and the jth coordinate of $F(v)$ is I_j. W number the unknowns (collected in the vector v) in the following way:

$$v_0 = a, \qquad v_1 = b, \qquad v_2 = |c|, \qquad v_k = x_{k-2} \qquad (3 \leqslant k \leqslant n+1).$$

Differentiating Eq. (16) we obtain a relation between the differentials of v and Y. Notation: The functional determinant of F is called J, the minor obtained from J by removing row i and column k is called $J_{i;k}$, while the subdeterminant obtained by removing rows i, j and columns k, l is called $J_{i,j;k,l}$.

To determine how, for fixed values of a, $Y_1, ..., Y_n$, the parameter b_1 varies as a function of Y_0, we use the first $n+1$ equations of the system (16). We put $dv_0 = dY_1 = \cdots = dY_n = 0$ and find

$$J_{n+1;0}\, dv_1 = J_{0,n+1;0,1}\, dY_0. \tag{17}$$

To determine, for fixed values of a, $Y_1, ..., Y_n$, the dependence of b_2 on Y_{n+1} we use the last $n+1$ equations of the systems (16). We find

$$J_{0;0}\, dv_1 = (-1)^n J_{0,n+1;0,1}\, dY_{n+1}. \tag{18}$$

The determinants are easily calculated by the method of [4]:
We introduce a measure on $\mathbb{R}$ with differential

$$dm(x) = e^{-ax^2 - bx} |x|^{m_0} |x-1|^{m_n} \prod_{k=1}^{n-1} |x - x_k|^{m_k - 1}\, dx. \tag{19}$$

For a finite-dimensional vector u with coordinates u_j we define the Vandermonde determinant

$$V(u) = \prod_{j<k} (u_k - u_j). \tag{20}$$

Using Eqs. (10), (12), and (19), we find for $j = 0, 1, ..., n+1$,

$$\begin{aligned} F_j(v) &= \int_{x_{j-1}}^{x_j} |c| \left(\prod_{k=1}^{n-1} |x - x_k| \right) dm(x) \\ &= s_j\, |c| \int_{x_{j-1}}^{x_j} \left(\prod_{k=1}^{n-1} (x - x_k) \right) dm(x), \end{aligned} \tag{21}$$

where

$$s_j = \begin{cases} (-1)^{n-1} & \text{for } j = 0 \\ (-1)^{n-j} & \text{for } j = 1, ..., n \\ 1 & \text{for } j = n+1. \end{cases} \tag{22}$$

The partial derivatives are

$$\frac{\partial F_j}{\partial v_0} = -s_j\, |c| \int_{x_{j-1}}^{x_j} x^2 \left(\prod_{q=1}^{n-1} (x - x_q) \right) dm(x) \tag{23$_0$}$$

$$\frac{\partial F_j}{\partial v_1} = -s_j\, |c| \int_{x_{j-1}}^{x_j} x \left(\prod_{q=1}^{n-1} (x - x_q) \right) dm(x) \tag{23$_1$}$$

$$\frac{\partial F_j}{\partial v_2} = s_j \int_{x_{j-1}}^{x_j} \left(\prod_{q=1}^{n-1} (x - x_q) \right) dm(x) \tag{23$_2$}$$

$$\frac{\partial F_j}{\partial v_{k+2}} = -s_j\, |c|\, m_k \int_{x_{j-1}}^{x_j} \left(\prod_{q \neq k} (x - x_q) \right) dm(x), \tag{23$_{k+2}$}$$

where the last formula is valid for $k = 1, ..., n-1$.

We start with

$$J_{0;0}=\left(\prod_{j=1}^{n+1}(-s_j)\right)|c|^n\left(\prod_{k=1}^{n-1}m_k\right)$$
$$\times\int_{x_0}^{x_1}\cdots\int_{x_n}^{x_{n+1}}\det(a_{jk})\,dm(u_{n+1})\cdots dm(u_1), \tag{24}$$

where, for $1\leqslant j\leqslant n+1$,

$$\begin{aligned} a_{j,1}&=u_j\prod_{q=1}^{n-1}(u_j-x_q)\\ a_{j,2}&=-\prod_{q=1}^{n-1}(u_j-x_q)\\ a_{j,k+2}&=\prod_{q\neq k}(u_j-x_q), \end{aligned} \tag{25}$$

where the last equation is valid for $k=1, ..., n-1$.

To simplify $\det(a_{jk})$ we subtract the last column from the previous $n-2$ columns. This permits us to extract the factor $\prod_{k=1}^{n-2}(x_k-x_{n-1})$. The new $n-1$ last columns are

$$a'_{j,n+1}=\prod_{q=1}^{n-2}(u_j-x_q)$$
$$a'_{j,k+2}=\prod_{\substack{q=1\\ q\neq k}}^{n-2}(u_j-x_q)\qquad (1\leqslant k\leqslant n-2).$$

Next, we subtract column n from the previous $n-3$ columns, which permits us to extract the new factor $\prod_{k=1}^{n-3}(x_k-x_{n-2})$. This goes on, until we get the matrix (b_{jk}), whose first two columns are those of (a_{jk}), while the $n-1$ last columns are

$$b_{j,k+2}=\prod_{q=1}^{k-1}(u_j-x_q)\qquad (1\leqslant k\leqslant n-1)$$

and the extracted factor $V(x)(-1)^{(n-1)(n-2)/2}$.

A new set of column operations permits us to eliminate the numbers x_q, so that we find $\det(b_{jk})=V(u)$. Thus, finally, using $\prod_{j=1}^{n+1}(-s_j)=(-1)^{(n-1)(n-2)/2}$, we obtain

$$J_{0;0}=|c|^n\left(\prod_{k=1}^{n-1}m_k\right)$$
$$\times\int_{x_0}^{x_1}\cdots\int_{x_n}^{x_{n+1}}V(x)\,V(u)\,dm(u_{n+1})\cdots dm(u_1). \tag{26}$$

Next,

$$J_{n+1;0}=\left(\prod_{j=0}^{n}(-s_j)\right)|c|^n\left(\prod_{k=1}^{n-1}m_k\right)$$
$$\times\int_{x_{-1}}^{x_0}\cdots\int_{x_{n-1}}^{x_n}\det(a_{jk})\,dm(u_n)\cdots dm(u_0), \tag{27}$$

where we again define the matrix elements a_{jk} by Eqs. (25), but this time for $j=0, ..., n$. Since $\prod_{j=0}^{n}(-s_j)=(-1)^{n(n-1)/2}$, we find

$$J_{n+1;0}=(-1)^{n-1}|c|^n\left(\prod_{k=1}^{n-1}m_k\right)$$
$$\times\int_{x_{-1}}^{x_0}\cdots\int_{x_{n-1}}^{x_n}V(x)\,V(u)\,dm(u_n)\cdots dm(u_0). \tag{28}$$

Note that the ranges of suffixes for $V(u)$ in Eqs. (26) and (28) are different.

Last,

$$J_{0,n+1;0,1}=\left(\prod_{j=1}^{n}(-s_j)\right)|c|^{n-1}\left(\prod_{k=1}^{n-1}m_k\right)$$
$$\times\int_{x_0}^{x_1}\cdots\int_{x_{n-1}}^{x_n}\det(a_{jk})\,dm(u_n)\cdots dm(u_1), \tag{29}$$

where we use only part of the matrix (a_{jk}), defined in Eqs. (25), viz. corresponding to row numbers in the range $[1, n]$ and column numbers in the range $[2, n+1]$. With this new definition we obtain $\det(a_{jk})=V(x)\,V(u)(-1)^{(n-2)(n-3)/2}$, while $\prod_{j=1}^{n}(-s_j)=(-1)^{n(n+1)/2}$, so that

$$J_{0,n+1;0,1}=(-|c|)^{n-1}\left(\prod_{k=1}^{n-1}m_k\right)$$
$$\times\int_{x_0}^{x_1}\cdots\int_{x_{n-1}}^{x_n}V(x)\,V(u)\,dm(u_n)\cdots dm(u_1). \tag{30}$$

To utilize these results we note that, as mentioned earlier, keeping Y_j $(1\leqslant j\leqslant n)$ fixed makes $c, x_1, ..., x_{n-1}$ holomorphic functions of a and b, so that also $I_0=I_0(a, b)$ and $I_{n+1}=I_{n+1}(a, b)$ become holomorphic functions of a and b. We have just shown that, for fixed a, I_0 is an increasing, I_{n+1} a decreasing function of b. This makes b_1 and b_2 well-defined functions of respectively Y_0 and Y_{n+1} for fixed a. But we are interested also in the dependence of b_1 and b_2 on a. We could explore this by looking at functional determinants, but it suffices for us to show that b_1 and b_2 are continuous functions of a. By definition, $I_0(a, b_1(a, Y_0)) = Y_0$. To show that b_1, for fixed Y_0, is a lower semicontinuous function of a, we must, for arbitrary constant b, show that the set $\{a \mid b_1(a, Y_0) > b\}$ is open. But this set equals the set $\{a \mid Y_0 > I_0(a, b)\}$, which is open because I_0 is continuous. Similarly, we can show that b_1 is upper semicontinuous, thus continuous, as a function of a. In the same way, b_2 is shown to be continuous as a function of a, for fixed Y_{n+1}.

Clearly, for a close to 0, we have $b_1 < 0 < b_2$. Next we set $b = -a$ and let a approach infinity, which makes I_{n+1}/I_S approach 0 and I_S/I_0 go toward infinity. Thus, in this limit, we have $b_2 < -a < b_1$. We conclude that for some positive value of a we have $b_1 = b_2$, showing the existence of a function f compatible with S.

(2) The set A has at most one extreme element. For definiteness, we take the case where A has no maximal element. Again there are two subcases:

(2a) A has a minimal element. Set $A^\circ = \mathbb{N}$. There are two possibilities:

(2a1) The value y_0 is infinite. Define the sequence of polynomials f_n $(n \geqslant m_1 + m_2)$ by

$$f_n'(z) = c_n \prod_{j=1}^{q} (1 - (z/x_j^{(n)}))^{m_j}, \tag{31}$$

where $x_1^{(n)} = x_1$ and $x_2^{(n)} = x_2$ independently of n (x_1 and x_2 are given positive numbers; for all j and n we shall have $x_j^{(n)} < x_{j+1}^{(n)}$), and for all j and n: $f_n(x_j^{(n)}) = y_j$. The degree n of f' equals $\sum_{j=1}^{q} m_j$. According to the theorem in [4] this determines f_n uniquely. We shall show that the sequence of polynomials f_n forms a normal family.

The first step is to show that the sum $\sum_{j=1}^{q} (m_j/x_j^{(n)})$ is bounded independently of n. Assume that for some sequence of q-values (where still $n = \sum_{j=1}^{q} m_j$) we have $\sum_{j=1}^{q} (m_j/x_j^{(n)}) \to \infty$. For $2 \leqslant j \leqslant q$, putting

$$U_j = (1/|c_n|) \int_{x_{j-1}^{(n)}}^{x_j^{(n)}} |f_n'(x)|\, dx, \tag{32}$$

we use

$$U_3/U_2 = |y_3 - y_2|/|y_2 - y_1|, \tag{33}$$

which is independent of n. Substituting the expression (31) into (32), dividing numerator and denominator in the resulting expression for U_3/U_2 by $\prod_{j=3}^{q} (1 - (x_2/x_j^{(n)}))^{m_j}$, and using the estimate

$$\log(1 - (x/x_j)) - \log(1 - (x_2/x_j)) \leqslant (x_2 - x)/(x_j - x_2) \tag{34}$$

(for $x_j > \max\{x, x_2\}$) in the numerator, while using the positivity of the left-hand side of Eq. (34) in the denominator (for $x < x_2 < x_j$), we obtain the upper bound

$$\frac{U_3}{U_2} \leqslant \frac{\int_{x_2}^{\infty} (x/x_1 - 1)^{m_1} (x/x_2 - 1)^{m_2} \exp(-(x - x_2) \sum_{j=3}^{q} (m_j/x_j^{(n)}))\, dx}{\int_{x_1}^{x_2} (x/x_1 - 1)^{m_1} (1 - x/x_2)^{m_2}\, dx}, \tag{35}$$

whose numerator tends toward 0 when q runs through the given sequence, while the denominator is constant. This contradicts Eq. (33).

Next we must show that the coefficients c_n are bounded. We have the condition

$$|c_n| \int_{x_1}^{x_2} \prod_{j=1}^{q} |1 - (x/x_j^{(n)})|^{m_j}\, dx = |y_2 - y_1|. \tag{36}$$

It suffices to show that the integral $(= U_2)$ has a lower bound independent of n. Here we use our result that for some finite positive number A, independent of n,

$$\sum_{j=1}^{q} (m_j/x_j^{(n)}) \leqslant A. \tag{37}$$

Choose $j = J$ as the lowest suffix such that $x_j^{(n)} \geqslant 2x_2$. Then, since $(\log(1 - x))/x$ is negative and decreases for x increasing from 0 to $\frac{1}{2}$, we have for $x_1 \leqslant x \leqslant x_2$ that

$$\sum_{j=J}^{q} m_j \log(1 - (x/x_j^{(n)})) \geqslant -A_1 x,$$

where $A_1 = (2 \log 2) A$. Because of Eq. (37), we have $\sum_{j=3}^{J-1} m_j \leqslant 2Ax_2$, so that

$$\sum_{j=3}^{J-1} m_j \log(1 - (x/x_j^{(n)})) \geqslant 2Ax_2 \log(1 - (x/x_2)).$$

We conclude that the integral in Eq. (36) has the lower bound

$$\int_{x_1}^{x_2} \left(\frac{x}{x_1} - 1\right)^{m_1} \left(1 - \frac{x}{x_2}\right)^{m_2 + 2Ax_2} e^{-A_1 x}\, dx,$$

a positive constant independent of n. Thus the numbers c_n are bounded, and we have

$$\begin{aligned} |f_n'(z)| &\leqslant |c_n| \prod_{j=1}^{q} (1 + (|z|/x_j^{(n)}))^{m_j} \\ &\leqslant (\sup |c_n|)\, e^{A|z|}, \end{aligned} \tag{38}$$

so that the functions f'_n are uniformly bounded on every compact subset of $\mathbb{C}$. The same is true for the functions f_n, and they form a normal family. Accordingly, we can extract a subsequence converging toward an entire function, whose derivative must satisfy the inequality (38), and which then is of order at most 1 and of finite type.

To show that for each fixed suffix j the numbers $x_j^{(n)}$ are bounded we first show that to each suffix j there is a positive number A_j, so that for all large n we have $\sum_{k=j+1}^{q} (m_k/x_k^{(n)}) > A_j$. Assume that for minimal j there were a subsequence of n-values for which $\sum_{k=j+1}^{q} (m_k/x_k^{(n)}) \to 0$. For these n-values $x_j^{(n)}$ would be bounded and $x_{j+1}^{(n)}$ tend toward infinity. But we could obtain a lower bound for U_{j+1} using estimates similar to those we used when considering U_2 above: Let the numbers A and B satisfy $(e+1)\sup x_j^{(n)} < A < B$. Then, for sufficiently large n,

$$U_{j+1} > \int_A^B \prod_{k=1}^{j} \left(\frac{x}{x_k^{(n)}} - 1\right)^{m_k} dx$$

$$> (B-A)\exp\left(\sum_{k=1}^{j} m_k\right),$$

which could become arbitrarily large, and this is incompatible with the fact that U_2 is bounded and U_{j+1}/U_2 constant. But then, for an arbitrary value of j,

$$U_{j+1} \leqslant \int_{x_j^{(n)}}^{\infty} \left(\prod_{k=1}^{j} \left(\frac{x}{x_k^{(n)}} - 1\right)^{m_k}\right) e^{-A_j x}\, dx.$$

If $x_j^{(n)}$ approached infinity, the right-hand side would tend toward 0.

We conclude that f' has infinitely many zeros x_j $(1 \leqslant j < \infty)$, each one being the limit of a subsequence of $(x_j^{(n)})$ (same j) and having the right multiplicity. The sum $\sum_{j=1}^{\infty} (m_j/x_j)$ converges. Since the function f' is real, it has the form

$$f'(z) = ce^{-bz} \prod_{j=1}^{\infty} \left(1 - \frac{z}{x_j}\right)^{m_j} \tag{39}$$

with real c and b. Furthermore, considering $\log(f'(x)/c)$ with $x<0$ and using Fatou's lemma or the fact the exponential factor dominates the canonical product [1, Theorem 2.10.13], we see that b must be non-negative. Clearly, f is compatible with the set S.

(2a2) The set A has no maximal element, but a minimal, and y_0 is finite. Now the function f' does not always have a canonical product of genus 0, and so we use the approximations

$$f'_n(z) = c_n e^{b_n z} \prod_{j=1}^{q} (1 - (z/x_j^{(n)}))^{m_j} \exp(m_j z/x_j^{(n)}), \tag{40}$$

where for all n and j we have $x_j^{(n)} < x_{j+1}^{(n)}$. Let $x_1^{(n)} = x_1 < 0$ and $x_2^{(n)} = x_2 > 0$ be given real numbers, and for all n and j (also $j = 0$), $f_n(x_j^{(n)}) = y_j$. According to earlier results (case (1b1) above) these conditions define f_n. To show that the functions f_n form a normal family we first prove that the sums $\sum_{j=1}^{q} (m_j/(x_j^{(n)})^2)$ are bounded. Again we define the integrals U_j ($j = 1, ..., q$) by Eq. (32). Note that with Eq. (40) the expression for U_j can be written in the form

$$U_j = \int_{x_{j-1}^{(n)}}^{x_j^{(n)}} F(x) \exp\left(b_n x + \sum_{k=3}^{q} m_k L(x/x_k^{(n)})\right) dx, \tag{41}$$

where

$$L(x) = x + \log(|1 - x|), \tag{42}$$

and

$$F(x) = \exp\left(\sum_{k=1}^{2} m_k L(x/x_k)\right) \tag{43}$$

is the n-independent part of the integrand.

In the following we shall need a few properties of the function L defined in Eq. (42). They are collected in

LEMMA 6. *The function $L(x) = x + \log(|1 - x|)$ is negative for $x < 0$ and for $0 < x < 1$. We have $L(0) = 0$, and $L(x) \to -\infty$ for $x \to 1$. The function $L(x)/x^2$ decreases from 0 to $-\infty$ when x increases from $-\infty$ to 1 (for $x = 0$ the function takes the value $-\frac{1}{2}$), increases from $-\infty$ to $\frac{1}{2}$ for x increasing from 1 to 2, and decreases again to 0 for $x \to \infty$. A difference $L(x) - L(y)$, where $(1 - x)(1 - y) > 0$, can be written as $(x - y)\zeta/(\zeta - 1)$, where ζ lies between x and y.*

Proof. This is elementary, and I apologize for leaving it to the reader.

Going back to the proof of the theorem we again use the fact that the ratio U_3/U_2 is independent of n (see Eq. (33)). We divide numerator and denominator by $\exp(b_n x_2 + \sum_{k=3}^{q} m_k L(x_2/x_k^{(n)}))$ and use in the numerator the estimate (see Lemma 6)

$$\begin{aligned} L(x/x_k) - L(x_2/x_k) &= -((x - x_2)/x_k)\,\zeta/(x_k - \zeta) \\ &\leqslant -(x - x_2)\, x_2/x_k^2, \end{aligned} \tag{44}$$

valid for $x_2 < x < x_k$ (ζ lies between x_2 and x). In the denominator (for

$0 < x < x_2$) the first member of Eq. (44) is positive. If $b_n \leqslant 0$, we find the upper bound

$$\frac{U_3}{U_2} \leqslant \frac{\int_{x_2}^{\infty} F(x) \exp((b_n - x_2 \sum_{k=3}^{q} (m_k/(x_k^{(n)})^2))(x - x_2))\, dx}{\int_0^{x_2} F(x) \exp(-b_n(x_2 - x))\, dx}, \quad (45)$$

which shows that in this case $\sum_{k=1}^{q} (m_k/(x_k^{(n)})^2)$ must be bounded.

If $b_n > 0$, consider the ratio $U_1/U_2 = |y_1 - y_0|/|y_2 - y_1|$. Divide the numerator and denominator by $\exp(b_n x_1 + \sum_{k=3}^{q} m_k L(x_1/x_k^{(n)}))$. For $x < x_1 < 0 < x_2 \leqslant x_k$ we use in the numerator the estimate (see Lemma 6)

$$L(x/x_k) - L(x_1/x_k) = ((x_1 - x)/x_k)\zeta/(x_k - \zeta)$$
$$\leqslant (-r)(x_1 - x)/x_k^2, \quad (46)$$

where $x < \zeta < x_1$, and $-r = x_1 x_2/(x_2 - x_1)$. In the denominator (for $x_1 < x < 0$) the left-hand side of Eq. (46) is positive. We find

$$\frac{U_1}{U_2} \leqslant \frac{\int_{-\infty}^{x_1} F(x) \exp((-b_n - r \sum_{k=3}^{q} (m_k/(x_k^{(n)})^2))(x_1 - x))\, dx}{\int_{x_1}^{0} F(x) \exp(b_n(x - x_1))\, dx}, \quad (47)$$

which tends toward 0 when $\sum_{k=3}^{q} (m_k/(x_k^{(n)})^2) \to \infty$. We conclude that whatever the behaviour of b_n, there must be a positive number A, so that for all q

$$\sum_{k=1}^{q} (m_k/(x_k^{(n)})^2) < A. \quad (48)$$

It also follows from Eqs. (45) and (47) that b_n must be bounded below and above, respectively. For all n, let $|b_n| < b_{\max}$. To show that $|c_n|$ is bounded above and below we use the equation

$$|c_n|\, U_2 = |y_2 - y_1|, \quad (49)$$

which shows that upper and lower bounds for U_2 suffice. A simple upper bound for U_2 is $\int_{x_1}^{x_2} \exp(b_{\max} |x|) F(x)\, dx$. Now, for $0 \leqslant x \leqslant \frac{1}{2}$ the ratio $L(x)/x^2$ is negative and decreasing and so has the minimal value $-(4 \log 2 - 2)$. To obtain a lower bound for U_2 we find a lower bound for $\sum_{j=3}^{q} m_j L(x/x_j^{(n)})$, valid for $0 < x < x_2$. Define $j = J$ as the lowest suffix for which $x_j^{(n)} \geqslant 2x_2$. Then $\sum_{j=J}^{q} m_j L(x/x_j^{(n)}) \geqslant -A_1 x^2$, where $A_1 = A(4 \log 2 - 2)$. Also,

$$\sum_{j=3}^{J-1} m_j L(x/x_j^{(n)}) \geqslant L(x/x_2) \sum_{j=3}^{J-1} m_j \geqslant 4A(x_2)^2\, L(x/x_2).$$

This gives

$$U_2 \geqslant \int_0^{x_2} F(x)\exp(-xb_{\max} + 4A(x_2)^2 L(x/x_2) - A_1 x^2)\,dx. \tag{50}$$

As we shall see, these results are sufficient to ensure that the functions f'_n, and so the functions f_n, are uniformly bounded on compact subsets of $\mathbb{C}$:

We have

$$\log|f'_n(z)| = \log|c_n| + b_n \Re(z) + \sum_{j=1}^{q} m_j \Re\left(\frac{z}{x_j^{(n)}} + \log\left(1 - \frac{z}{x_j^{(n)}}\right)\right), \tag{51}$$

and only the last sum remains to be estimated. Consider the function

$$F(\mu) = r\mu + \tfrac{1}{2}\log(1 - 2r\mu + r^2) \qquad \text{for} \quad -1 \leqslant \mu \leqslant 1.$$

Here $r = |z/x_j^{(n)}|$, and $r\mu = \Re(z/x_j^{(n)})$.

Maximizing F means to maximize term number j in the sum of Eq. (51) for a fixed value of $|z|$. We find that

$$F'(\mu) = r^2 \frac{r - 2\mu}{1 - 2r\mu + r^2}.$$

For $r \leqslant 2$ we obtain the maximum $F(r/2) = r^2/2$. For $r > 2$ the maximum is $F(1) = r + \log(r-1) < 2(r-1)$. Define $j = J$ as the lowest suffix such that $x_j^{(n)} \geqslant |z|/2$. Then

$$\begin{aligned}\sum_{j=1}^{q} m_j \Re\left(\frac{z}{x_j^{(n)}} + \log\left(1 - \frac{z}{x_j^{(n)}}\right)\right) &\leqslant 2|z| \sum_{j=1}^{J-1} \frac{m_j}{x_j^{(n)}} + \sum_{j=J}^{q} m_j \frac{|z|^2}{2(x_j^{(n)})^2} \\ &\leqslant A|z|^2, \end{aligned} \tag{52}$$

which gives the required bound.

Arguing as in connection with the inequality (38) we find that from the functions f_n we can extract a subsequence, which converges toward an entire function f compatible with S:

To show that for each fixed suffix j the numbers $x_j^{(n)}$ are bounded we first show that to each suffix j there is a positive number A_j, so that for all $n > \sum_{k=1}^{j+1} m_k$ we have

$$\sum_{k=j+1}^{q} (m_k/(x_k^{(n)})^2) > A_j. \tag{53}$$

Assume that for some minimal j there is a subsequence of n-values for which $\sum_{k=j+1}^{q} (m_k/(x_k^{(n)})^2) \to 0$. For these n-values $x_j^{(n)}$ will be bounded and $x_{j+1}^{(n)}$ tend toward infinity.

Assume also that for a subsequence $b_n + \sum_{k=1}^{j} (m_k/x_k^{(n)}) \geq 0$. Then a lower bound for U_{j+1} can easily be derived (compare with the corresponding estimate in subcase (2a1)). Let the numbers A and B satisfy $(e+1)\sup x_j^{(n)} < A < B$. Then, for sufficiently large n,

$$U_{j+1} > \int_A^B \exp\left(\sum_{k=2}^{j} m_k - \kappa x^2 \sum_{k=j+1}^{q} (m_k/(x_k^{(n)})^2)\right) dx,$$

where the constant κ can be chosen arbitrarily close to (but greater than) $\frac{1}{2}$. Since the right-hand side of this inequality can be arbitrarily large, we have a contradiction.

If, on the other hand, for a subsequence, $b_n + \sum_{k=1}^{j} (m_k/x_k^{(n)}) < 0$, we can obtain a lower bound for U_1,

$$U_1 > \int_A^{x_1} \left(\frac{x}{x_1} - 1\right)^{m_1} \exp\left(-\frac{x^2}{2} \sum_{k=j+1}^{q} (m_k/(x_k^{(n)})^2\right) dx,$$

valid for arbitrary $A < x_1 < 0$. As again the right-hand side can become arbitrarily large, we have a contradiction.

But if an inequality (53) is valid, we obtain the upper bound

$$U_{j+1} < \int_{x_j^{(n)}}^{\infty} \left(1 - \frac{x}{x_1}\right)^{m_1} \left(\frac{x}{x_2} - 1\right)^{\sum_{k=2}^{j} m_k}$$
$$\times \exp\left(\left(b_{\max} + \frac{m_1}{x_1} + \frac{1}{x_2} \sum_{k=2}^{j} m_k\right) x - \frac{1}{2} A_j x^2\right),$$

which clearly tends toward 0, if $x_j^{(n)} \to \infty$ for $n \to \infty$.

Thus, the underlying assumption, that for some j the sequence $(x_j^{(n)})$ is unbounded, is false.

But then f' has infinitely many zeros, is of order at most 2 and of finite type, with canonical product of genus at most 1, and so of the form (1). Here we can look at $\log(c/f'(x)) + bx$ for $x < 0$ and conclude, by means of Fatou's lemma or [1, Theorem 2.10.13], that $a \geq 0$.

(2b) The final case, where we prove existence of a function f compatible with the given set S, is the one where the ordered set $A = A^\circ$ has no extreme elements. We can put $A = \mathbb{Z}$. We approximate f by functions f_p (with $p \in \mathbb{N}$) of the type from Eq. (39) with $f_p(-\infty)$ infinite and

$$f_p(x_j^{(p)}) = y_j \tag{54}$$

for all integers $j > -p$ (i.e., the set A corresponding to f_p is $[-p, \infty) \cap \mathbb{Z}$). The numbers $x_0 < 0$ and $x_1 > 0$ are fixed, and $x_j^{(p)} < x_{j+1}^{(p)}$ for $j > -p$. Since we cannot expect the genus of the canonical product of f' to be 0, we write Eq. (39) in the form

$$f_p'(z) = c_p e^{b_p z} \prod_{k=-p+1}^{\infty} ((1-(z/x_k^{(p)}))^{m_k} \exp(m_k z/x_k^{(p)})). \tag{55}$$

For each p, as noted earlier, we have $b_p + \sum_{k=-p+1}^{\infty} (m_k/x_k^{(p)}) \leqslant 0$. We keep the definition (42) and define, similarly to Eq. (43),

$$F(x) = \exp\left(\sum_{k=0}^{1} m_k L(x/x_k)\right) \tag{56}$$

for the p-independent part of the integrand in the expressions Eq. (32) for U_j $(j > -p+1)$. The things to prove and the arguments with which to prove them are very similar to those of the previous case. Some complication is inevitable: In every integrand we have both positive and negative values of $x_k^{(p)}$. Thus we need the estimates (44) and (46) with suffixes reduced by one (and so also $-r = x_0 x_1/(x_1 - x_0)$) to evaluate terms with $k > 1$, but also, to evaluate terms with $k < 0$, the inequality (44) with x_2 replaced by x_0 and valid for $x_k < x < x_0 < 0$, while, for $x_k < x_0 < 0 < x_1 < x$, the inequality (46) is replaced by

$$\begin{aligned} L(x/x_k) - L(x_1/x_k) &= -((x-x_1)/x_k)\,\zeta/(x_k - \zeta) \\ &\leqslant (-r)(x-x_1)/x_k^2, \end{aligned} \tag{57}$$

where $x_1 < \zeta < x$, and, as before, $-r = x_0 x_1/(x_1 - x_0)$.

We find, corresponding to Eq. (45), for $b_p \leqslant 0$,

$$\begin{aligned} U_2/U_1 \leqslant \int_{x_1}^{\infty} & F(x) \exp\left(\left(b_p - x_1 \sum_{k=2}^{\infty} (m_k/(x_k^{(p)})^2)\right)(x - x_1)\right) \\ & \times \exp\left(\left(-r \sum_{k=-p+1}^{-1} (m_k/(x_k^{(p)})^2)\right)(x-x_1)\right) dx \Bigg/ \\ & \int_0^{x_1} F(x) \exp(-b_p(x_1 - x))\, dx, \end{aligned} \tag{58}$$

which shows that in this case $\sum_{k=-p+1}^{\infty} (m_k/(x_k^{(p)})^2)$ is bounded independently of p. For $b_p > 0$ we use

$$\begin{aligned} U_0/U_1 \leqslant \int_{-\infty}^{x_0} & F(x) \exp\left(\left(-b_p + x_0 \sum_{k=-p+1}^{-1} (m_k/(x_k^{(p)})^2)\right)(x_0 - x)\right) \\ & \times \exp\left(\left(-r \sum_{k=2}^{\infty} (m_k/(x_k^{(p)})^2)\right)(x_0 - x)\right) dx \Bigg/ \\ & \int_{x_0}^{0} F(x) \exp(b_p(x - x_0))\, dx, \end{aligned} \tag{59}$$

showing that also in this case $\sum_{k=-p+1}^{\infty} (m_k/(x_k^{(p)})^2)$ is bounded, i.e., there is a positive number A, so that

$$\sum_{k=-p+1}^{\infty} (m_k/(x_k^{(p)})^2) < A. \tag{60}$$

It also follows from the inequalities (58) and (59) that there is a positive number $b_{\max}$, so that

$$|b_p| \leqslant b_{\max} \tag{61}$$

for all p. Obviously, $U_1 \leqslant \int_{x_0}^{x_1} F(x) \exp(b_{\max} |x|)\, dx$, so that $|c_p|$ has a lower bound. Finally, using Lemma 6,

$$\begin{aligned} U_1 \geqslant \int_0^{x_1/2} F(x)\, exp\left(-xb_{\max} - (x^2/2) \sum_{k=-p+1}^{-1} (m_k/(x_k^{(p)})^2)\right) \\ \times \exp\left(4L\left(\frac{1}{2}\right) x^2 \sum_{k=2}^{\infty} (m_k/(x_k^{(p)})^2)\right) dx, \end{aligned} \tag{62}$$

which, together with Eq. (60), gives a positive lower bound for U_1, and thus a finite upper bound for $|c_p|$.

We finally need a proof that for each suffix j the sequence $(x_j^{(p)} \mid p \in \mathbb{N})$ is bounded. To show this we first investigate how a function f_p changes when one of its stationary values is altered.

Assume that f is a function of the type described in the theorem, and let r be a simple zero of f'. Assume that we want to change the stationary value $s = f(r)$ but keep the others, and also keep the two abscissas x_0 and x_1. Let us try the variation

$$f_\varepsilon(x) = f(x) + \varepsilon f'(x) \frac{(x - x_0)(x - x_1)}{x - r}. \tag{63}$$

We have $f_\varepsilon(r) = f(r) + \varepsilon f''(r)(r - x_0)(r - x_1)$; the change (of the first order in ε) of s is, in fact, $\varepsilon f''(r)(r - x_0)(r - x_1)$, and we can show that

$$\frac{\partial f}{\partial s} = \frac{f'(x)(x - x_0)(x - x_1)}{f''(r)(r - x_0)(r - x_1)(x - r)}.$$

The variation (63) generates a transformation of f into a function whose stationary values are the same as those of f (with the same multiplicities), except for s, which is changed by the amount specified above. To obtain this function we generate a sequence $f^{(N)}$ $(N = 1, 2, ...)$ of functions, where for each N the function $f^{(N)}$ is obtained by applying the variation (63), with ε replaced by ε/N, N times in succession (i.e., $f^{(2)} = f_{\varepsilon/2, \varepsilon/2}$, etc.). Then $f^{(N)}$ approaches a limit (uniformly on compact subsets of $\mathbb{C}$), which is the function we look for. There are a number of tedious details in arguments of this kind: for instance, f'_ε has all zeros real if $|\varepsilon|$ is sufficiently small; this is easily seen to be correct if f is a polynomial, and the permissible magnitude of $|\varepsilon|$ is determined only by the behaviour of f in a neighbourhood of r, not by its asymptotic behaviour or by the degree of the polynomial. It is well known that a function f' of the Pólya–Laguerre class can be obtained as the limit of polynomials with only real zeros, and the argument thus carries over. The variation (63) partially splits up multiple zeros of f'. The corresponding stationary values are altered only by $o(\varepsilon)$, which is the reason for the ultimate convergence of the sequence $f^{(N)}$; also, splitting up of r occurs if we try to use (63) on functions f with r being a multiple zero of f'. We can then define a transformation where the various split-up zeros are used succesively as r-values to change a stationary value of multiplicity greater than one.

The great advantage with these transformations is the ease with which the associated change of value of various functionals can be predicted (this was noted previously by Vladimir Markov; see [6]). For instance, if we consider the variation (63), we see that, except for points in the immediate vicinity of $x=r$, the graph of f_ε at a point x is moved horizontally by the amount $-\varepsilon(x-x_0)(x-x_1)/(x-r)+o(\varepsilon)$ relative to the graph of f. This is the information we need for f_p: The transition from f_{p+1} to f_p can be performed by first (using a finite number of arbitrarily small variations) splitting up the zero $x_{-p}^{(p+1)}$ of f'_{p+1}, next moving the zero of smallest absolute value in the resulting cluster to $-\infty$ by increasing the absolute value of the corresponding stationary value to infinity. This is done by variations of type Eq. (63) with $\varepsilon<0$ (here, as before, a process of going to the limit is involved). This will give a horizontal movement of the graph of f_{p+1} (outside the interval $[x_0, x_1]$) away from r. So the sequence $(x_j^{(p)} \mid p\in \mathbb{N})$ is decreasing and therefore convergent for all $j>1$. To handle the case where $j<0$ we note that from f_p we can go back to a polynomial ($P_{p,q}$, say) with the first $q-1$ correct stationary values to the right of the origin by increasing $|y_q|$ to infinity, which will further enhance a possible divergence $x_j^{(p)} \to -\infty$ for $p\to\infty$. But then we can use the existence of a function (f_{-q}, say) with all stationary values y_j $(j<q)$ correct and $f_{-q}(\infty)$ infinite, from which $P_{p,q}$ can be obtained by increasing $|y_{-p}|$ to infinity, which for any p, increases $x_j^{(p)}$, and we have a contradiction. We conclude that we can extract a convergent subsequence from the sequence (f_p), that the limit function f has a derivative f' with infinitely many negative (and positive) zeros x_j, and that f is compatible with S. Moreover, f is of order at most two, and f' has the form specified in Eq. (1). Here a must be non-negative, since otherwise $|f'(z)|$ would tend toward 0 for $z\to\infty$ along the imaginary axis, which is in conflict with the known behaviour of the functions f'_p.

We have concluded the proof of the theorem.

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REFERENCES

1. R. P. BOAS, "Entire Functions," Academic Press, New York, 1954.
2. G. M. GOLUZIN, "Geometric Theory of Functions of a Complex Variable," Amer. Math. Soc., Providence, RI, 1969.
3. BO KJELLBERG, On integral functions bounded on a given set, *Mat. Tidsskrift B* (1952), 92–99.
4. G. K. KRISTIANSEN, Characterization of polynomials by means of their stationary values, *Arch. Math. (Basel)* **43** (1984), 44–48.
5. G. K. KRISTIANSEN, Characterization of real entire functions by means of their stationary values, *Arch. Math. (Basel)* **56** (1991), 278–280.
6. V. MARKOV, Über Polynome, die in einem gegebenen Intervall möglichst wenig von Null abweichen, *Math. Ann.* **77** (1916), 213–258.
7. G. PÓLYA AND G. SZEGÖ, "Problems and Theorems in Analysis," Springer-Verlag Berlin/Heidelberg/New York, 1972.